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New $\mathcal{N} = 2$ superconformal field theories from M/F theory orbifolds

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Abstract

We consider M-theory on $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n$ with M5 branes wrapped on \mathbf{R}^2 . One can probe this background with M5 branes wrapped on \mathbf{T}^2 . The theories on the probes provide many new examples of $\mathcal{N} = 2$ field theories without Lagrangian description. All these theories have Coulomb branches, and we find the corresponding Seiberg-Witten curves. The exact solution is encoded in a Hitchin system on an orbifolded torus with punctures. The theories we consider also arise from D3 probes in F-theory on $K3 \times K3$ orbifolds. Interestingly, the relevant F-theory background has frozen \mathbf{Z}_n singularities which are analogous to frozen \mathbf{Z}_2 singularities in Type I string theory. We use the F-theory description to find supergravity duals of the probe SCFT's in the large N limit and compute the spectrum of relevant and marginal operators. We also explain how the decoupling of $U(1)$ factors is manifested in the supergravity description.

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1 Introduction

In the last few years many examples of nontrivial superconformal field theories (SCFT) in $d = 4$ have been found. In particular, there exists a uniform way to construct $\mathcal{N} = 2$ SCFT's by looking for singular points on the Coulomb branch of $\mathcal{N} = 2$ field theories. Although we do not yet have a complete understanding of these SCFT's, one can learn a lot about them from the low-energy effective action away from the singular points (i.e. from the Seiberg-Witten solution) [1]. In addition, when an SCFT has a large N limit, it can often be described by IIB supergravity on $AdS^5 \times X$ for a suitable X [2, 3].

In this paper we are going to describe and study a class of $\mathcal{N} = 2$ SCFT's which arise from consideration of M5 probes on M-theory orbifolds. The M-theory background we consider is $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n$, $n = 2, 3, 4, 6$, where the \mathbf{Z}_n action preserves sixteen supersymmetries. For $n = 3, 4, 6$ the complex structure of \mathbf{T}^2 must be restricted appropriately. We will think of $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n$ as an elliptic fibration over $\mathbf{R}^2/\mathbf{Z}_n$. The M-theory background will also include k M5 branes wrapping the noncompact base of this fibration. The resulting configuration has eight unbroken supersymmetries. The probe M5's are wrapping the elliptic fiber at N points of the base. We will find it convenient to work on the n -fold cover of the base \mathbf{R}^2 . Then, counting images, we have nN probe M5's wrapped at nN points of \mathbf{R}^2 . The probes do not break supersymmetry further.

For a \mathbf{Z}_2 orbifold ($n = 2$) the $\mathcal{N} = 2$ theory on the probes has a Lagrangian description. To see this, one takes the τ -parameter of the fiber to infinity. In this limit the M-theory background described above reduces to a IIA configuration containing two $O6^-$ planes, with two D6 branes on top of each of them, and k NS5 branes. (The reason one gets two D6 branes on top of each of the $O6^-$ planes is that in such a configuration the Ramond-Ramond charge is cancelled locally. Hence this IIA configuration lifts to a locally flat M-theory geometry, namely an orbifold geometry.) The probe M5 branes reduce to D4 branes suspended between NS5 branes. Such “elliptic” brane configurations have been previously considered in [4]. Elliptic brane configurations do not allow for the bending of NS5 branes and therefore always produce theories with vanishing beta-functions [5]. In the present case the low-energy theory on D4 branes has a product gauge group with both symplectic and unitary factors [4]. The gauge groups and matter content are the same as in $d = 6$ theories arising on the worldvolume of Type I D5

branes near a \mathbf{Z}_{2k} singularity [6].

When $n = 3, 4, 6$ the τ -parameter of the fiber is a root of unity, so no IIA description exists. As a consequence, the probe theories do not have a Lagrangian description. However, by taking the volume of the fiber to zero, one may pass to an F-theory description. The dual F-theory background is $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n \times \mathbf{C}^2/\mathbf{Z}_{nk}$, and the probe M5 branes become probe D3 branes. Note that the F-theory background can be regarded as an orbifold limit of $K3 \times K3$, therefore it has eight unbroken supersymmetries. Introduction of D3 probes does not break supersymmetry further. This F-theory background can be also thought of as a collection of several mutually nonlocal 7-branes wrapped on $\mathbf{C}^2/\mathbf{Z}_{nk}$. The IIB coupling is constant over the base and equal to a root of unity. The case $k = 0$ is especially simple since the 7-branes become flat. In this case the theory on the 7-branes has a gauge group E_6, E_7 , or E_8 depending on the value of n [7]. This implies that the theory on the probe D3 branes has an exceptional global symmetry and does not admit a Lagrangian description. The latter presumably remains true for $k > 0$. Thus we obtain three infinite families of new $\mathcal{N} = 2$ SCFT's without Lagrangian description labeled by $n = 3, 4, 6$.

An important subtlety is that the F-theory orbifold corresponding to M-theory on $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n$ with k M5 branes is not a geometric orbifold. By this we mean that some of the blow-up modes of the $\mathbf{C}^2/\mathbf{Z}_{nk}$ orbifold are absent. Concretely, we will show that the \mathbf{Z}_{nk} singularity on which the 7-branes are wrapped can only be resolved to a product of k \mathbf{Z}_n singularities. These “frozen” \mathbf{Z}_n singularities cannot be further resolved. For $n = 2$ the “frozen” singularity is T-dual to the “frozen” singularity of Type I on $\mathbf{C}^2/\mathbf{Z}_2$ [8]. For $n > 2$ we obtain a nonperturbative generalization of [8]. This is discussed in more detail in section 3.

To study new SCFT's we will make use both of M-theory and F-theory descriptions. The M-theory setup is convenient for finding Seiberg-Witten solutions. In the next section we show that for all n and k the solution is encoded in a Hitchin system on an orbifolded torus with punctures. From this we derive the Seiberg-Witten curve (the spectral cover of the Hitchin system) and the S-duality group of the probe theories. As a matter of fact, the Hitchin system solves not just the theory in $d = 4$, but its compactification on a circle of arbitrary radius [9]. The F-theory description, discussed in detail in section 3, is helpful for finding supergravity duals of the SCFT's in the limit of large number of probes. In section 4 we use these supergravity duals to compute the spectrum of relevant and marginal operators in the SCFT's. Most of the

spectrum can be checked independently using our knowledge of the Seiberg-Witten solution, along the lines of [1]. We find complete agreement between the two approaches. We also discuss the decoupling of $U(1)$ factors in the boundary gauge theory (in the cases where the boundary SCFT is equivalent to a gauge theory). We show that from the supergravity point of view this effect arises from a certain subtlety in the Kaluza-Klein reduction of a 2-form on $AdS^5 \times S^1$. The details of this argument are explained in the Appendix. Our conclusions are summarized in section 5.

2 Seiberg-Witten curves from M theory orbifolds

Consider M-theory on $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n$ with k M5 branes wrapped on $\mathbf{R}^2/\mathbf{Z}_n$ and N M5 branes wrapped on the elliptic fiber \mathbf{T}^2 at N points of $\mathbf{R}^2/\mathbf{Z}_n$ (or nN points of \mathbf{R}^2 , if we work on the cover). n is one of the integers 2, 3, 4, 6. For $n = 3, 6$ the τ -parameter of \mathbf{T}^2 has to be $\exp(i\pi/3)$, for $n = 4$ $\tau = i$, and for $n = 2$ τ is unrestricted. If we denote the complex affine coordinates on \mathbf{T}^2 and \mathbf{R}^2 by z and v , then the \mathbf{Z}_n action is

$$z \rightarrow \omega z, \quad v \rightarrow \omega^{-1} v, \quad (1)$$

where $\omega = \exp(2\pi i/n)$. This orbifolding preserves sixteen supersymmetries. The M5 branes wrapped on \mathbf{R}^2 break half of the remaining supersymmetries. The probe M5 branes wrapping \mathbf{T}^2 do not break any further supersymmetries. To distinguish the two sets of M5 branes we will call the branes wrapping \mathbf{R}^2 the M5' branes.

2.1 \mathbf{Z}_2 orbifold

We start with brane configurations which have a IIA limit. The corresponding probe theories are described by $\mathcal{N} = 2$ gauge theories in the infrared. The IIA limit exists when the τ -parameter of the torus can be taken to infinity, so we have to set $n = 2$. Let the two coordinates on \mathbf{T}^2 be x^6, x^{10} , and the coordinates on \mathbf{R}^2 be x^4, x^5 . The probe M5 branes wrap x^6, x^{10} , while M5' branes wrap x^4, x^5 . Upon reduction to IIA the orbifold background in M-theory reduces to $(\mathbf{R}^2 \times \mathbf{S}^1)/(\Omega(-1)^{F_L} \mathcal{R}_{456})$, where S^1 is parametrized by x^6 , Ω is worldsheet parity, and \mathcal{R}_{456} is the reflection of x^4, x^5, x^6 . The action

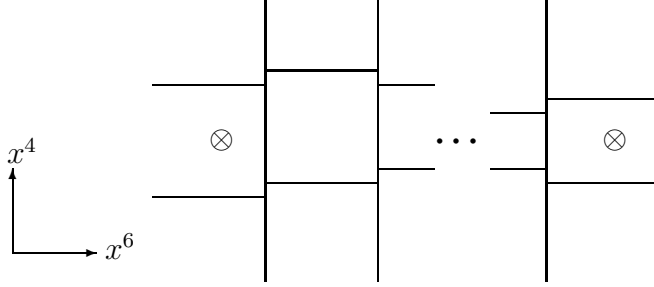


Figure 1: Elliptic IIA brane configuration. Vertical and horizontal lines are NS5 branes and D4 branes, respectively. \otimes denotes an $O6^-$ plane with two D6 branes on top. Only half of the circle parametrized by x^6 is shown.

of \mathcal{R}_{456} has two fixed planes: $x^4 = x^5 = x^6 = 0$ and $x^4 = x^5 = 0, x^6 = \pi R_6$. These fixed planes are the $O6^-$ planes of IIA. There are also two D6 branes on top of each of the orientifold planes.¹ The M5 branes reduce to $2N$ D4 branes stretched along x^6 . The M5' branes reduce to k NS5 branes extended in x^4, x^5 and localized in x^6 . The resulting brane configuration is shown in Figure 1. It appeared previously in [4]. It preserves eight supersymmetries, therefore the theory on D4 branes is an $\mathcal{N} = 2$ $d = 4$ theory at low energies. Well-known arguments [5] show that its gauge group \mathcal{G} is $Sp(N) \times SU(2N)^{k-1} \times Sp(N)$. Naively, the gauge group is $Sp(N) \times U(2N)^{k-1} \times Sp(N)$, but the $U(1)$ factors decouple [5]. To state the matter content, we arrange the simple factors of \mathcal{G} along a line. There is a bifundamental hypermultiplet for every pair of neighboring group factors. The two $Sp(N)$ factors in addition have two fundamentals each. The resulting gauge theory has zero beta-functions and is an example of a finite $\mathcal{N} = 2$ theory. We will show later that this theory has an interesting S-duality group. Theories in $d = 6$ with precisely this gauge group and matter content have previously appeared in the study of D5 probes near a \mathbf{Z}_{2k} singularity in Type I [6, 11]. This is not a coincidence: T-duality along x^6, x^4, x^5 converts our setup to that studied in [6, 11].

It is also possible to put unpaired NS5 branes on top of orientifold planes. The resulting brane configurations are shown in Figures 2 and 3. In Figure 2, we have $2k$ pairs of NS5 branes and one unpaired NS5 brane. The latter can move in the x^7, x^8, x^9 but not in x^6, x^{10} . The theory on D4 branes

¹To see this consider a D2 brane probing this background. The theory on the D2 brane located near one of the $O6^-$ planes is an $\mathcal{N} = 4$ $d = 3$ $SU(2)$ theory with two fundamentals. Its moduli space is an orbifold $(\mathbf{R}^3 \times \mathbf{S}^1)/\mathbf{Z}_2$ [10]. Therefore the corresponding M-theory background is also a \mathbf{Z}_2 orbifold.

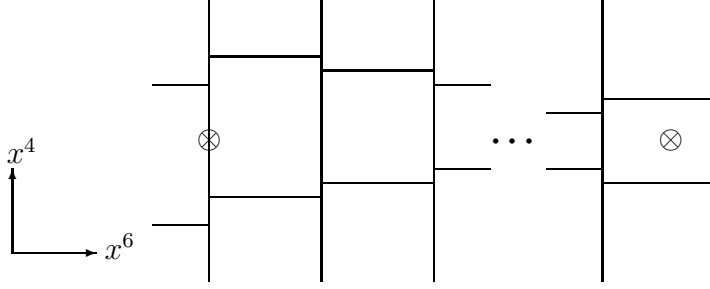


Figure 2: Elliptic IIA brane configuration with an unpaired NS5 brane on top of one of the orientifold planes. The notation is the same as in Figure 1.

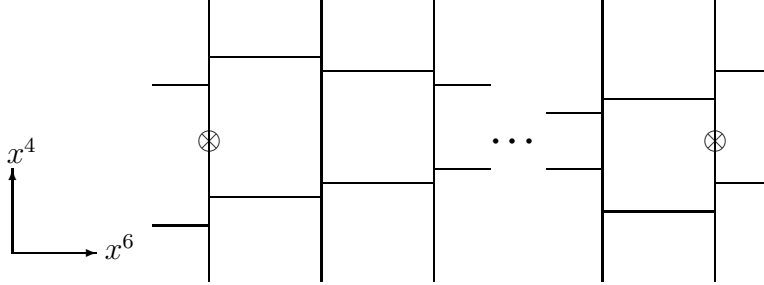


Figure 3: Elliptic IIA brane configuration with two unpaired NS5 branes. The notation is the same as in Figure 1.

has gauge group $SU(2N)^k \times Sp(N)$. There are bifundamental hypers for each pair of neighboring simple factors. The first $SU(2N)$ factor has in addition a hypermultiplet in the two-index antisymmetric representation and two fundamentals, and the $Sp(N)$ factor has two fundamentals. In Figure 3, we have $2k$ paired NS5 branes and two unpaired ones at each of the orientifold planes. The gauge group is $SU(2N)^{k+1}$, with k bifundamentals. The first and last factor also have an antisymmetric tensor and two fundamentals. It is easy to check that all these gauge theories are finite.

Flavor symmetries acting on the fundamental hypermultiplets are $(Spin(4) \times Spin(4))/\mathbf{Z}_2$, $(U(2) \times Spin(4))/\mathbf{Z}_2$, and $(U(2) \times U(2))/\mathbf{Z}_2$ for configurations in Figures 1, 2, and 3, respectively. The \mathbf{Z}_2 quotient arises because part of the naive flavor symmetry is gauged. There is also a global $U(1)$ for each bifundamental and antisymmetric tensor. The case $k = 0$ is special, since the gauge group becomes simple. Indeed, for $k = 0$ both Figures 1 and 2 correspond to an $Sp(N)$ gauge theory with an antisymmetric tensor

and four fundamentals, while Figure 3 yields an $SU(2N)$ gauge theory with two antisymmetric tensors and four fundamentals. Correspondingly, the flavor symmetry acting on the fundamentals is enhanced to $Spin(8)/\mathbf{Z}_2$ in the former case and to $U(4)/\mathbf{Z}_2$ in the latter.

The way we constructed our brane configuration does not allow to have nonzero masses for fundamental hypermultiplets. These masses become nonzero if the D6 branes do not coincide with $O6^-$ planes. However in such a configuration the Ramond-Ramond charge is no longer cancelled locally. The corresponding M-theory background is curved and is not described by an orbifold. On the other hand, the masses for the bifundamentals and the antisymmetric tensors are allowed by the orbifold geometry. They correspond to the differences in the center-of-mass positions of the neighboring stacks of D4 branes [5]. Because of this, the sum of all the masses must be zero: $\sum_{\alpha} m_{\alpha} = 0$. As in [5], this condition can be relaxed by introducing a shift in x^4, x^5 as one goes around x^6 . The corresponding mass parameter is referred to as the global mass [5, 4].

Let us consider the effect of T-duality along x^6 . The $O6^-$ planes and four D6 branes become an $O7^-$ plane and four D7 branes, with a $Spin(8)/\mathbf{Z}_2$ gauge theory on their worldvolume. This worldvolume is not flat: four of its coordinates are wrapped on an ALF space with an orbifold singularity resulting from T-dualizing NS5 branes. For configurations in Figures 1, 2, and 3 this singularity is \mathbf{Z}_{2k} , \mathbf{Z}_{2k+1} , and \mathbf{Z}_{2k+2} , respectively. There is a subtlety here: as noticed by Polchinski [8], in the presence of an orientifold projection there is a possibility of having frozen \mathbf{Z}_2 singularities, with no blow-up modes. This phenomenon is analogous to discrete torsion in oriented string theory [12]. As a matter of fact, many blow-up modes *are* frozen in our situation. Namely, the configuration T-dual to Figure 1 has only enough blow-up modes for a resolution $\mathbf{Z}_{2k} \rightarrow (\mathbf{Z}_2)^k$, the one T-dual to Figure 2 can be blown-up as $\mathbf{Z}_{2k+1} \rightarrow (\mathbf{Z}_2)^k$, and the one T-dual to Figure 3 as $\mathbf{Z}_{2k+2} \rightarrow (\mathbf{Z}_2)^k$. To see this, note that the blow-up modes correspond to the motion of NS5 branes in x^7, x^8, x^9 . However, an NS5 brane and its mirror image always move together, so the corresponding \mathbf{Z}_2 singularity cannot be resolved. The unpaired NS5 branes can move independently. This simple observation leads immediately to the “freezing” pattern described above.

A “frozen” \mathbf{Z}_2 singularity can be “thawed” by bringing an NS5 brane and its image to an orientifold plane and then separating them in the x^7, x^8, x^9 . After this procedure we loose the zero mode corresponding to the motion along x^6, x^{10} , but gain a zero mode corresponding to the separation of the

NS5 branes in x^7, x^8, x^9 . At the transition point the NS5 brane and its image are coincident, so there are tensionless strings. After T-duality in x^6 this phase transition can be described as follows. We are dealing with D7 branes wrapped on a “frozen” $\mathbf{C}^2/\mathbf{Z}_2$ orbifold. The low-energy $d = 4$ theory is lacking a blow-up mode (a hypermultiplet), but has an extra vector multiplet. We will call this branch of the moduli space the Coulomb branch. At a special point on the Coulomb branch we may pass to a Higgs branch, where the vector multiplet becomes massive and the blow-up mode reappears. The $d = 6$ version of this transition was considered in [11, 13]. In particular, [13] gives a description of the $d = 6$ transition in terms of a brane configuration which is T-dual to ours. We will discuss these issues in more detail in section 3.

Let us now obtain the Seiberg-Witten solution for these $\mathcal{N} = 2$ theories. We start with the theory in Figure 1. The method was explained in detail in [9]. We compactify x^3 on a circle of radius R and perform the 3 – 10 flip. The brane configuration now consists of $2N$ D4 branes wrapping $\mathbf{T}^2/\mathbf{Z}_2$ and k D4′ branes piercing them at k points p_1, \dots, p_k . It is more convenient to think of D4 branes as wrapped on the double cover \mathbf{T}^2 , with $2k$ punctures located symmetrically with respect to a fixed point of the involution. We will use an affine parameter z as a coordinate on \mathbf{T}^2 . The theory on D4 branes is an impurity theory [14] whose Higgs branch is the mirror of the Coulomb branch of the original theory compactified on a circle. This Higgs branch is the moduli space of $U(2N)$ Hitchin equations with residues [14]

$$F_{z\bar{z}} - [\Phi_z, \Phi_z^\dagger] = 0, \\ \bar{D}\Phi_z = -\frac{\pi}{RR_6} \sum_{\alpha=1}^k (\delta^2(z - z_\alpha) + \delta^2(z + z_\alpha)) \text{diag}(m_\alpha, 0, \dots, 0). \quad (2)$$

The parameters m_α are related to the masses of the bifundamentals. Taking the trace of the second equation in Eq. (2) one can see that m_α satisfy $\sum_\alpha m_\alpha = 0$. The Higgs field Φ describes the positions of the D4 branes in x^4, x^5 , so it has to satisfy

$$\Phi(z) = -M\Phi(-z)M^{-1}. \quad (3)$$

Here M describes the action of the orbifold group on the Chan-Paton factors of the D4 branes. According to [6] (see also [15]) M generates a regular

representation of \mathbf{Z}_2 [6]:

$$M = \begin{pmatrix} 1_{N \times N} & 0 \\ 0 & -1_{N \times N} \end{pmatrix} \quad (4)$$

Similarly, the $U(2N)$ gauge field A_z must satisfy

$$A_z(z) = -MA_z(-z)M^{-1}. \quad (5)$$

The moduli space of solutions of Hitchin equations with these constraints has a natural hyperkähler metric. This metric is the exact metric on the Coulomb branch of the $d = 4$ $\mathcal{N} = 2$ theory compactified on a circle of radius R . It is a very complicated metric, but things get simpler in the decompactification limit $R \rightarrow \infty$. In this limit one is interested in the Seiberg-Witten curve, which is the spectral cover of the Hitchin system [16, 9]. The spectral cover can be constructed without actually solving Hitchin equations. By definition, it is a $2N$ -fold cover of the elliptic curve $\Sigma = \mathbf{T}^2$ on which the Hitchin system lives given by the equation

$$\det(v - \Phi(z)) = 0. \quad (6)$$

Explicitly, the spectral cover is given by

$$v^{2N} + f_1 v^{2N-1} + f_2 v^{2N-2} + \cdots + f_{2N} = 0, \quad (7)$$

where f_1, \dots, f_{2N} are meromorphic functions on Σ , by virtue of Hitchin equations. They have simple poles at the points $z_1, \dots, z_k, -z_1, \dots, -z_k$. Furthermore, Eq. (3) implies that f_{2j} are even functions of z , while f_{2j-1} are odd. Let us represent Σ as a cubic curve

$$y^2 = (x - e_1)(x - e_2)(x - e_3). \quad (8)$$

The above conditions constrain f_ℓ to be of the form

$$\begin{aligned} f_{2\ell} &= \sum_{\alpha=1}^k \frac{a_{\ell\alpha}}{x - x_\alpha} + a_{\ell 0}, \\ f_{2\ell-1} &= \sum_{\alpha=1}^k \frac{y b_{\ell\alpha}}{x - x_\alpha}, \quad \sum_{\alpha} b_{\ell\alpha} = 0, \end{aligned} \quad (9)$$

where $a_{\ell 0}, \dots, a_{\ell k}$ and $b_{\ell 1}, \dots, b_{\ell k}$ are complex constants, $x_\alpha = \mathcal{P}(z_\alpha)$, and $\mathcal{P}(z)$ is the Weierstrass elliptic function. Thus the spectral cover depends on

$2kN$ parameters. Some of them are the Coulomb branch moduli, and some are the parameters of the theory, namely the masses of the bifundamentals m_α . As in [5], the mass parameters determine the asymptotic behaviour of the noncompact curve Σ . Therefore they must be identified with the residues of f_1 . Alternatively, it is easy to see from Hitchin equations that the residues of f_1 are proportional to m_α . Since $\sum_\alpha m_\alpha = 0$, we have $k - 1$ mass parameters. All the other parameters of the curve are moduli. Their number, $2kN - k + 1$, agrees with the dimension of the Coulomb branch expected from field theory.

Let us now turn to the brane configurations in Figures 2 and 3. The only difference compared to the previous case is that the Higgs field is allowed to have residues at some of the fixed points of Σ . For the theory in Figure 2 the Seiberg-Witten curve is given by the same equations Eqs. (7,8,9), except that now the parameters $b_{\ell\alpha}$ need not satisfy the constraint $\sum_\alpha b_{\ell\alpha} = 0$. The total number of parameters is $(2k + 1)N$, out of which k are mass parameters and $2kN + N - k$ are moduli, in agreement with field theory. For the theory in Figure 3 the functions f_ℓ are given by

$$f_{2\ell} = \sum_{\alpha=1}^k \frac{a_{\ell\alpha}}{x - x_\alpha} + a_{\ell 0},$$

$$f_{2\ell-1} = \sum_{\alpha=1}^k \frac{yb_{\ell\alpha}}{x - x_\alpha} + \frac{yb_{\ell 0}}{x - e_3}.$$

The number of moduli is $(2N - 1)(k + 1)$, while the number of mass parameters is $k + 1$. This agrees with the field theory count.

We mentioned above that the restriction on the masses of the bifundamentals $\sum_\alpha m_\alpha = 0$ can be removed by introducing a shift along x^4, x^5 as one goes around x^6 [5]. A way of introducing this deformation into the Hitchin equations was explained in [9]: one simply replaces $\text{diag}(m_\alpha, 0, \dots, 0)$ in Eq. (2) with $\text{diag}(m_\alpha, -M, \dots, -M)$. Then one can easily see that the constraint becomes $\sum_\alpha m_\alpha = (2N - 1)kM$. We will not discuss the corresponding modification of the Seiberg-Witten curve, but we will keep in mind that the total number of mass parameters is $k, k + 1$, and $k + 2$ for the theories in Figures 1, 2, and 3, respectively.

Given Seiberg-Witten curves it is easy to derive the S-duality group of the theories. In all three cases the gauge couplings and theta-angles are determined by the τ -parameter of Σ and the location of the punctures. Let us denote by $\mathcal{M}_{p,k}$ the moduli space of an orbifolded elliptic curve Σ/\mathbf{Z}_2 ,

with k marked points not coinciding with the orbifold points, and with p orbifold points marked as well ($p = 0, 1, 2$). Then the S-duality groups of the theories in Figures 1, 2, and 3 are the fundamental groups of $\mathcal{M}_{0,k}$, $\mathcal{M}_{1,k}$, and $\mathcal{M}_{2,k}$, respectively.

If we set all mass parameters to zero and go to the origin of the moduli space, we obtain an $\mathcal{N} = 2$ superconformal field theory. It has a number of deformations which preserve supersymmetry, and from the curve we can read off their R-charges and dimensions. Since the deformation parameters are chiral primary fields which are $SU(2)_R$ singlets, their dimensions and R-charges are related by $\Delta = R/2$ [1].

First, there are exactly marginal deformations obtained by varying the gauge couplings and theta-angles of the theory. They have zero R-charge and dimension. They are encoded in the locations of the punctures and the complex structure of Σ . Their total number is $k + 1$. Second, there are Coulomb branch moduli and the masses of the bifundamentals. The R-charge of the moduli can be determined from the curve as follows. The R-symmetry is realized in the brane configuration as a rotation in the x^4, x^5 plane, and the standard normalization is such that the Higgs field has R-charge 2. This means that v in Eq. (7) has R-charge 2. As for x and y , they have zero R-charge. This determines the R-charges of all parameters in Eq. (7). Namely, $a_{\ell\alpha}$ has R-charge 4ℓ , and $b_{\ell\alpha}$ has R-charge $4\ell - 2$. In particular the masses (i.e. $b_{1\alpha}$) have R-charge 2. This is the expected result: the masses are the lowest components of background vector multiplets which couple to conserved currents, therefore their R-charge is 2 in any $\mathcal{N} = 2$ theory. Since the deformation parameters are chiral primary fields which are $SU(2)_R$ singlets, their dimensions and R-charges are related by $\Delta = R/2$ [1].

As mentioned above, it is impossible to introduce masses for fundamental hypermultiplets without making the M-theory background curved. Our solution is only valid when these masses are zero. Still, we know on general grounds that these deformations exist, and their R-charge is 2.

2.2 \mathbf{Z}_n orbifolds for $n > 2$

We now consider M-theory on the orbifold $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n$ for $n = 3, 4, 6$. The τ -parameter of the torus is $\exp(i\pi/3)$ or i depending on whether $n = 3, 6$ or $n = 4$. The configuration also includes k M5' wrapping the base of this fibration and N M5 branes wrapping the fiber. These configurations do not have a IIA reduction. Nevertheless we can still find the Seiberg-Witten

solution for the theory on the M5 branes using the method of [9], i.e. by compactifying x^3 and performing the 3 – 10 flip. We obtain nN D4 branes wrapping $\mathbf{T}^2/\mathbf{Z}_n$ with k punctures. On the n -fold cover of \mathbf{T}^2 we thus have nk punctures located at $z = \omega^j z_\alpha, \alpha = 1, \dots, k, j = 0, \dots, n-1$. Here $\omega = \exp(2\pi i/n)$. The Higgs branch of the corresponding impurity theory is described by $U(nN)$ Hitchin equations of the form

$$F_{z\bar{z}} - [\Phi_z, \Phi_z^\dagger] = 0, \quad \bar{D}\Phi_z = -\frac{\pi}{RR_6} \sum_{\alpha=1}^k \text{diag}(m_\alpha, 0, \dots, 0) \sum_{j=0}^{n-1} \delta^2(z - \omega^j z_\alpha). \quad (10)$$

The Higgs field and the gauge connection are in the adjoint of $U(nN)$. They have to satisfy

$$\Phi(\omega z) = \omega^{-1} M \Phi(z) M^{-1}, \quad A_z(\omega z) = \omega^{-1} M A_z(z) M^{-1}, \quad (11)$$

where

$$M = 1_{N \times N} \otimes \text{diag}(1, \omega, \dots, \omega^{n-1}) \quad (12)$$

generates the regular representation of \mathbf{Z}_n . The trace of the second equation in Eq. (10) implies $\sum_\alpha m_\alpha = 0$. As before, this condition can be relaxed by introducing an analogue of the global mass, namely by replacing $\text{diag}(m_\alpha, 0, \dots, 0)$ in Eq. (10) by $\text{diag}(m_\alpha, -M, \dots, -M)$. Then m_α satisfy $\sum m_\alpha = (nN-1)kM$. We do not consider this modification in what follows.

Another interesting modification is to introduce punctures at the fixed points of the orbifolded torus. This amounts to having residues for Φ at the fixed points. It is easy to see that this is consistent with the projection Eq. (11). Recall that a puncture away from the fixed point together with its $n-1$ images corresponds to an M5' brane wrapping the base $\mathbf{R}^2/\mathbf{Z}_n$. Then a puncture sitting at a fixed point of the orbifold must correspond to a $1/n^{\text{th}}$ of the usual M5' brane. Although this is an interesting possibility, we will not consider it here.

The Hitchin system defined above provides a solution for the probe theories compactified on a circle of radius R . The Seiberg-Witten curve is the spectral cover of the Hitchin system. It has the form

$$v^{nN} + f_1 v^{nN-1} + f_2 v^{nN-2} + \dots + f_{nN} = 0. \quad (13)$$

The functions f_ℓ are meromorphic functions with simple poles at nk points $z = \omega^j z_\alpha$. By virtue of Eq. (11) they satisfy

$$f_\ell(\omega z) = \omega^{-\ell} f_\ell(z). \quad (14)$$

These conditions completely determine the Seiberg-Witten curve.

As an example, let us work out the explicit form of the curve for $n = 3$. It will be obvious then how to extend the discussion to larger n . The elliptic curve with a \mathbf{Z}_3 automorphism can be thought of as a cubic curve $y^2 = x^3 - 1$. The solution of Eq. (14) looks as follows:

$$f_\ell = \sum_{\alpha=1}^k a_{\ell\alpha} (y + y_\alpha) \sum_{j=0}^2 \frac{\omega^{-\ell j}}{\omega^{2j} x - x_\alpha} + b_\ell \sum_{j=0}^2 \omega^{\ell j}. \quad (15)$$

Here $x_\alpha = \mathcal{P}(z_\alpha)$, $y_\alpha = \mathcal{P}'(z_\alpha)/2$, $a_{\ell\alpha}$ and b_ℓ are complex constants. Note that the last term in Eq. (15) is nonvanishing only if $\ell = 0 \bmod 3$. In addition the coefficients $a_{\ell\alpha}$ must satisfy a constraint which follows from the requirement that f_ℓ 's be nonsingular at $x = \infty$:

$$\sum_{\alpha=1}^k a_{\ell\alpha} \sum_{j=0}^2 \omega^{-j(\ell+2)} = 0.$$

This constraint is nontrivial only if $\ell = 1 \bmod 3$. Now we can count the number of parameters in the equation of the curve. Let us denote the dimension of the space of f_ℓ 's satisfying Eq. (14) by d_ℓ . Then we get

$$d_\ell = \begin{cases} k + 1, & \ell = 0 \bmod 3 \\ k - 1, & \ell = 1 \bmod 3 \\ k, & \ell = 2 \bmod 3 \end{cases} \quad (16)$$

It follows that the total number of parameters is $3kN$. The R-charges of the parameters can be read off the curve in the same way as in the previous subsection: the parameters entering f_ℓ have R-charge 2ℓ . Since we have no Lagrangian description of the theory, we have nothing to compare these results with. Nevertheless, we can determine which of these parameters are moduli and which are “masses.” The $k - 1$ residues of f_1 determine the asymptotic behaviour of the curve, so they are the “masses.” Alternatively, it is easy to see from Hitchin equations that these residues are proportional to m_α 's which are the parameters of the system. Their R-charge is 2, as is appropriate for mass parameters. They may be regarded as living in vector multiplets. (It is on these grounds that we call them masses.) One should not forget about the possibility of introducing the global mass M (see above). Taking it into account, the total number of background vector multiplets is

k . The possibility to couple k vector multiplets implies that the theory in question has k conserved $U(1)$ currents. Of course, the actual current algebra may be bigger than that. Recall that our curve for the \mathbf{Z}_2 orbifold did not contain masses for the fundamentals and therefore missed flavor symmetries acting on them. It seems likely that the same is true for other orbifolds. In the next section we will discuss this question from the point of view of F-theory. It will be shown that for $n = 2$ and $k = 0$ (no punctures) the theory has a E_6 current algebra (in addition to the $U(1)$'s found above), while for $k > 0$ it has some subalgebra of E_6 .

Similar analysis can be performed for $n = 4$ and $n = 6$. Here we just state the results. For all n we find

$$d_\ell = \begin{cases} k + 1, & \ell = 0 \bmod n \\ k - 1, & \ell = 1 \bmod n \\ k & \text{otherwise} \end{cases} \quad (17)$$

The total number of parameters in the equation of the curve is nkN . Out of these k are “masses” (including the global mass) and the rest are moduli. Thus for any n we have k $U(1)$ currents. It will be shown in the next section that for $k = 0$, in addition to these $U(1)$ currents, the $n = 4$ and $n = 6$ theories have E_7 and E_8 current algebras, respectively. For $k > 0$ certain subalgebras of these current algebras remain.

Finally, let us determine the S-duality groups. We have k exactly marginal deformations related to the location of the punctures. Therefore the S-duality group is the fundamental group of the moduli space of an orbifolded elliptic curve Σ/\mathbf{Z}_n with k marked points.

3 F-theory duals

In the limit when the volume of the elliptic fiber goes to zero, M-theory on $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n$ is equivalent to F-theory on the same manifold. This manifold can be regarded as an orbifold limit of a “noncompact K3.” Since the τ -parameter of \mathbf{T}^2 is constant, we get an F-theory background with constant coupling [17, 7]. For $n = 2$ this background is nothing but a IIB orientifold background $\mathbf{R}^2/(\Omega(-1)^{F_L} \mathcal{R}_{45})$ with four D7 branes on top of the fixed point. Here \mathcal{R}_{45} is the reflection of x^4, x^5 . (We could also obtain this result by first taking a IIA limit of the M-theory configuration and then T-dualizing

along x^6 .) For $n = 3, 4, 6$ the F-theory backgrounds can be thought of as a collection of several mutually nonlocal 7-branes in IIB. In all four cases the theory living on the worldvolume of the 7-branes has nonabelian gauge group G . For $n = 2, 3, 4, 6$ this group is $Spin(8)/\mathbf{Z}_2, E_6, E_7, E_8$, respectively.

The M-theory background studied in the previous section also contained nk M5' branes wrapping the base \mathbf{R}^2 and nN probe M5 branes wrapping \mathbf{T}^2 . Let us first consider the case $k = 0$, when M5' branes are absent. The probe M5 branes turn into N D3 branes parallel to x^0, x^1, x^2, x^3 . For $n = 2$ the theory on D3 branes is an $\mathcal{N} = 2$ $Sp(N)$ gauge theory with an antisymmetric tensor and four fundamentals, in agreement with IIA arguments [18]. For $n = 3, 4, 6$ the probe theory does not have a Lagrangian description. For any n the 7-brane gauge symmetry G is the global symmetry of the probe theory. Thus for $n = 3, 4, 6$ the probe theory has an E_6, E_7, E_8 current algebra, respectively. This justifies the claims made in the previous section.

Now let us consider the case $k > 0$. In M-theory we have nk M5' branes, counting images. In F-theory they become an orbifold $\mathbf{C}^2/\mathbf{Z}_{nk}$. Here \mathbf{C}^2 has coordinates x^6, x^7, x^8, x^9 , and therefore the orbifold plane is parallel to $x^0, x^1, x^2, x^3, x^4, x^5$. Thus the full F-theory background is $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n \times \mathbf{C}^2/\mathbf{Z}_{nk}$. It can be thought of as an orbifold limit of $K3 \times K3$ near the orbifold singularity.

The blow-up modes of the $\mathbf{C}^2/\mathbf{Z}_{nk}$ orbifold correspond to the motion of M5' branes along x^7, x^8, x^9 . However there are only k such modes, since an M5' brane and its $n - 1$ images move together. This means that the \mathbf{Z}_{nk} singularity can only be resolved to a product of k \mathbf{Z}_n singularities, most of the blow-up modes being “frozen.” We may call this branch of vacua the Coulomb branch. Instead of the blow-up modes one has k extra complex scalars which are the lowest components of $d = 4$ $\mathcal{N} = 2$ vector multiplets. Each such scalar corresponds to the x^6, x^{10} position of an M5' brane on the M-theory side. As in the \mathbf{Z}_2 case, the blow-up modes can be “thawed” by passing to a Higgs branch. To this end one must tune the above mentioned complex scalar to a particular value. At this point the missing $n - 1$ blow-up modes become massless, and one can give them VEVs. On the M-theory side this corresponds to bringing an M5' brane and its $n - 1$ images to a fixed point of the orbifold and then separating them in the x^7, x^8, x^9 .

Because of the freezing of the blow-up modes, the F-theory background is not really a geometric orbifold $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n \times \mathbf{C}^2/\mathbf{Z}_{nk}$. A similar phenomenon occurs in Type I theory on $\mathbf{C}^2/\mathbf{Z}_2$. It was shown by Polchinski [8] that there are two different orientifolding procedures for Type IIB on $\mathbf{C}^2/\mathbf{Z}_2$.

The first procedure is simply to quotient by worldsheet parity Ω . Then, as is common in perturbative constructions, there is a trapped flux of the B-field through the shrunk 2-cycle, implying that the $Spin(32)/\mathbf{Z}_2$ bundle on the 9-branes has a nontrivial generalized Stiefel-Whitney class \tilde{w}_2 [19, 20]. The 2-cycle can be blown up, and then one discovers a $Spin(32)/\mathbf{Z}_2$ instanton sitting where the fixed point used to be [19]. This instanton breaks $Spin(32)/\mathbf{Z}_2$ down to $U(16)/\mathbf{Z}_2$. This type of \mathbf{Z}_2 singularity arises in the Gimon-Polchinski model [21].

The second procedure is to quotient by ΩJ . Here J is the symmetry of the worldsheet conformal field theory of $\mathbf{C}^2/\mathbf{Z}_2$ which flips the sign of the twisted sector. The projection ΩJ kills the zero mode responsible for the blow-up, keeping instead a $(1,0)$ tensor multiplet. Also, it does not have a trapped flux of the B-field. As in the previous case tadpole cancellation requires 16 9-branes carrying an $Spin(32)/\mathbf{Z}_2$ bundle. It also requires this bundle to have a nontrivial monodromy \mathcal{M} when the singular point of $\mathbf{C}^2/\mathbf{Z}_2$ is deleted. (\mathcal{M} is encoded in a nontrivial action of the orbifold group on the Chan-Paton factors). The monodromy breaks $Spin(32)/\mathbf{Z}_2$ down to $(Spin(16) \times Spin(16))/\mathbf{Z}_2$. This second orientifold is not a geometric orbifold of Type I, because it lacks a blow-up mode. However, one can pass to a geometric phase by tuning the real scalar in the tensor multiplet, so that tensionless strings arise. At this point one can go to the Higgs branch, where the tensor multiplet is lifted, and the singularity is resolved. This transition was described in detail in [11, 13]. Unlike the Coulomb branch (i.e. the branch with the frozen \mathbf{Z}_2 singularity), the Higgs branch cannot be realized by a free CFT.

Guided by these Type I considerations one can easily guess the right orbifold action in our case. In fact, as remarked in section 2, for $n = 2$ our brane configuration is T-dual to that in [13], and therefore also to that in [8]. It follows that one can obtain the right orbifold action simply by T-dualizing the construction in [8] along two directions parallel to the orbifold plane (x^4, x^5 in our notation). It is also easy to guess how to generalize to $n \neq 2$. One should start with F-theory on

$$\mathbf{T}^2 \times \mathbf{R}^2 \times \mathbf{C}^2/\mathbf{Z}_{nk},$$

where the complex structure of \mathbf{T}^2 is fixed appropriately. Then one should orbifold further by $\mathcal{P}_n J_n$, where \mathcal{P}_n is the generator of the SUSY-preserving \mathbf{Z}_n action on $\mathbf{T}^2 \times \mathbf{R}^2$, and J_n multiplies the twisted sectors of the \mathbf{Z}_{nk} orbifold

by phases. It is the presence of J_n which distinguishes this construction from a geometric orbifold. To describe J_n precisely it is convenient to think of $nk - 1$ twisted sectors as arising from $nk - 1$ shrunk 2-cycles. They are acted upon by a regular representation of \mathbf{Z}_{nk} . One can choose the basis of the 2-cycles so that the generator of \mathbf{Z}_{nk} multiplies the s^{th} sector by $\exp(2\pi i s/nk)$. In this basis J_n acts by multiplying the s^{th} sector by ω^s , where $\omega = \exp(2\pi i/n)$. In other words, J_n generates a natural \mathbf{Z}_n subgroup of \mathbf{Z}_{nk} . For $n = 2$ it is easy to see that this construction is T-dual to that in [8] and produces frozen $\mathbf{C}^2/\mathbf{Z}_2$ singularities (k of them). It is natural to expect that in general one gets k frozen $\mathbf{C}^2/\mathbf{Z}_n$ singularities, because the above “orientifolding” procedure makes sense only when \mathbf{Z}_n singularities are present, and not on a resolved ALE space.

Next we have to figure out the $d = 4$ gauge group H due 7-branes. It becomes part of the global symmetry of the D3 probes. For $k = 0$ H is the same as the eight-dimensional gauge group, i.e. $H = Spin(8)/\mathbf{Z}_2, E_6, E_7$, or E_8 , depending on the value of n . But for $k > 0$ the vector bundle on the 7-branes can have a nontrivial monodromy \mathcal{M} when the singular point of $\mathbf{C}^2/\mathbf{Z}_{nk}$ is removed. \mathcal{M} breaks the eight-dimensional gauge group G down to a subgroup $H \subset G$. We can be slightly more specific about H . Since the fundamental group of $\mathbf{C}^2/\mathbf{Z}_{nk}$ with the origin removed is \mathbf{Z}_{nk} , \mathcal{M} generates at most a \mathbf{Z}_{nk} subgroup. In fact, we expect \mathcal{M} to generate a \mathbf{Z}_n subgroup, since the \mathbf{Z}_{nk} singularity can be resolved to a product of k \mathbf{Z}_n singularities with identical monodromies. Unfortunately there are many inequivalent choices of \mathcal{M} resulting in different H . Thus additional constraints are required.

The problem of computing H is part of a more general problem, namely understanding the twisted sectors of the $\mathcal{P}_n J_n$ projection. For $n = 2$ the twisted sectors are simply open strings, and their choice is constrained by perturbative consistency conditions of [21, 8]. These conditions uniquely fix $H = (Spin(4) \times Spin(4))/\mathbf{Z}_2$. This is most easily seen by T-dualizing the set-up of [8]. The monodromy is realized as the action of the \mathbf{Z}_2 orbifold group on the Chan-Paton factors. Tadpole cancellation requires \mathcal{M} to be of the form

$$\mathcal{M} = \begin{pmatrix} 1_{4 \times 4} & 0 \\ 0 & -1_{4 \times 4} \end{pmatrix}. \quad (18)$$

Clearly, it breaks $Spin(8)/\mathbf{Z}_2$ down to $(Spin(4) \times Spin(4))/\mathbf{Z}_2$. Not surprisingly, H agrees with the flavor symmetry of the gauge theory on the probe branes (see section 2).

For $n > 2$ twisted states include multi-prong strings [22]. It is far from clear what replaces the conditions of [21, 8] in this case. It would be very interesting to pin down the exact content of the twisted sector of the F-theory orbifold which corresponds to the M-theory configuration of section 2. In any case, twisted states must include a vector multiplet in the adjoint of H .

A peculiarity of the $n = 2$ case is that the eight-dimensional gauge group $G = Spin(8)/\mathbf{Z}_2$ is not simply connected. Thus one may also consider a \mathbf{Z}_2 orbifold with a nontrivial generalized Stiefel-Whitney class. Such a \mathbf{Z}_2 singularity is T-dual to that constructed by Gimon and Polchinski [21] and therefore can be blown up. The corresponding M-theory configuration has two unpaired NS5 brane at the orientifold planes, as in Figure 3. For $n > 2$ the gauge group is simply connected, so no analogue of the Stiefel-Whitney class exists. Still, one can have a configuration analogous to Figure 3, namely n M5' branes stuck at n fixed points of the \mathbf{Z}_n orbifold. It is not clear to us what this configuration maps to in F-theory. In what follows we will not consider F-theory configurations corresponding to M-theory backgrounds with stuck M5' branes.

4 The large N limit

When the number N of three-branes at the singularity $(\mathbf{T}^2 \times \mathbf{R}^2)/\mathbf{Z}_n \times \mathbf{C}^2/\mathbf{Z}_{nk}$ becomes large, $\mathcal{N} = 2$ superconformal field theory on the brane world-volume is expected to be "holographic" to the near horizon limit of a supergravity solution [2]. In our case this near horizon limit is $AdS^5 \times S^5/(\mathbf{Z}_{nk} \times \mathbf{Z}_n)$. To describe the $\mathbf{Z}_{nk} \times \mathbf{Z}_n$ action we consider the $SU(2)_L \times SU(2)_R \times U(1)_R$ subgroup of $SU(4)$ which is a cover of the isometry group of S^5 . \mathbf{Z}_{nk} is embedded in the $SU(2)_L$ factor. The generator of \mathbf{Z}_n looks like $\mathcal{P}_n J_n$, where \mathcal{P}_n acts as a \mathbf{Z}_n automorphism on the \mathbf{T}^2 of F-theory and simultaneously as a $U(1)_R$ rotation, while J_n multiplies the twisted sectors of the \mathbf{Z}_{nk} orbifold by phases, as described in section 3. This orbifolding reduces supersymmetry from sixteen to eight supercharges and breaks $SU(4)$ down to $SU(2)_R \times U(1)_R \times U(1)_L$. The $SU(2)_R \times U(1)_R$ becomes the R-symmetry of the SCFT on the boundary, while $U(1)_L$ becomes a global non-R symmetry. We will identify this $U(1)_L$ symmetry more precisely later.

According to the AdS/SCFT correspondence [3] conformal dimensions of operators in the boundary SCFT are related to masses of the supergravity

states. For a p-form state this relation is

$$m^2 = (\Delta - p)(\Delta + p - 4). \quad (19)$$

In this section we compute the supergravity spectrum and compare it with the results of section 2. We consider only states which couple to relevant and marginal operators. We will show that there is complete agreement between the AdS approach and the the approach based on the analysis of the Seiberg-Witten curve.

Before plunging into computational details let us make a few general comments. Due to supersymmetry it is sufficient to consider only bosonic states. Supersymmetry also protects the dimensions of the operators and the masses of the supergravity modes from α' corrections as long as they come in short multiplets. In particular, the dimension of a chiral primary operator is determined solely by its $U(1)_R$ charge and the dimension d of its $SU(2)_R$ representation:

$$\Delta = \left\lfloor \frac{R}{2} \right\rfloor + d - 1. \quad (20)$$

One can divide supergravity states into three different groups according to their origin. First, there are states coming from projecting the supergravity spectrum on $AdS^5 \times S^5$. We will call them bulk modes. Second, there are twisted states of the \mathbf{Z}_{nk} orbifold which are invariant with respect to $\mathcal{P}_n J_n$. Third, there are states twisted with respect to $\mathcal{P}_n J_n$. They are charged with respect to the 7-brane gauge group H . We consider these three groups of states in turn.

4.1 Bulk supergravity states

The spectrum of the bulk modes can be obtained by decomposing $SU(4)$ representations of supergravity on $AdS_5 \times S^5$ into representations of $SU(2)_L \times SU(2)_R \times U(1)_R$, and projecting onto states invariant under the orbifold action. The \mathbf{Z}_{nk} action is embedded into the $SU(2)_L$ factor, while the geometric part of the \mathcal{P}_n action is embedded into $U(1)_R$. The commutant of \mathbf{Z}_{nk} in $SU(2)_L$ is the $U(1)_L$ symmetry corresponding to a non-R symmetry on the boundary. The projection on the invariant states proceeds in two steps. In the first step we project on the \mathbf{Z}_{nk} -invariant states. The states with zero $U(1)_L$ charge are automatically \mathbf{Z}_{nk} invariant. Every odd-dimensional representation of $SU(2)_L$ contains exactly one such state, while even dimensional

representations contain none [23]. More generally, a state is \mathbf{Z}_{nk} invariant if and only if its $U(1)_L$ charge is a multiple of nk . Both even-dimensional and odd-dimensional representations of $SU(2)_L$ may contain such states. In the second step we require that the states be invariant with respect to $\mathcal{P}_n J_n$. For states which originate from the $d = 10$ fields other than the B -fields this simply means that the R-charge is a multiple of $2n$ [24, 25]. Special attention should be paid to the modes coming from the reduction of the B -fields, since the nongeometric part of \mathcal{P}_n induces non-trivial monodromy on these fields [24]. After one diagonalizes the monodromy matrix one finds that the selection rule for these modes is $R = \pm 2 \bmod 2n$. It is sufficient to consider only the plus sign, since the other choice of sign is obtained by complex conjugation.

The supergravity spectrum on $AdS^5 \times S^5$ has been worked out in [26]. Supergravity states (and hence the SCFT operators) organize themselves into multiplets of $SU(2, 2|2)$. Below we discuss bosonic states which survive the orbifold projection, limiting ourselves to those states which couple to relevant or marginal operators. We first discuss the states with zero $U(1)_L$ charge, and then make some comments about states which transform nontrivially under $U(1)_L$.

The graviton is a singlet of $SU(4)$ and therefore is in the spectrum of any $\mathbf{Z}_{nk} \times \mathbf{Z}_n$ orbifold. It couples to the stress-energy tensor (marginal operator of dimension 4). The spectrum of supergravity on $AdS^5 \times S^5$ also contains a massless gauge boson in the adjoint (**15**) of $SU(4)$. A decomposition of **15** into irreps of $SU(2)_L \times SU(2)_R \times U(1)_R$ yields $\mathbf{15} = (\mathbf{1}, \mathbf{1})_0 + (\mathbf{2}, \mathbf{2})_2 + (\mathbf{2}, \mathbf{2})_{-2} + (\mathbf{3}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3})_0$. The projection on the \mathbf{Z}_{nk} -invariant states leaves $\mathbf{1}_0 + \mathbf{1}_0 + \mathbf{3}_0$, which is automatically \mathbf{Z}_n -invariant. The $\mathbf{3}_0$ and one of the $\mathbf{1}_0$ gauge bosons couple to the $SU(2)_R$ and $U(1)_R$ currents on the boundary, respectively. The remaining massless vector is a $U(1)_L$ gauge boson which couples to a non-R current on the boundary. In section 2 we found that the boundary SCFT has k $U(1)$ currents. The current coupled to the $U(1)_L$ gauge boson is one of them. (We will make this identification more precise below). The remaining $k - 1$ currents of the SCFT must couple to gauge bosons which come from the twisted sectors.

Now let us turn to 2-form states. They satisfy first-order equations of the antiself-dual type (and their complex conjugates satisfy equations of the self-dual type) [26, 27]. The ones which couple to relevant or marginal operators satisfy $m^2 = 1$ and $m^2 = 4$ and transform as **6** and **20'** of $SU(4)$, respectively. These 2-forms originate from the B -fields in $d = 10$, so the selection rule for

their R-charges is $R = 2 \bmod 2n$. $\mathbf{6}$ decomposes with respect to $SU(2)_L \times SU(2)_R \times U(1)_R$ as $(\mathbf{2}, \mathbf{2})_0 + (\mathbf{1}, \mathbf{1})_2 + (\mathbf{1}, \mathbf{1})_{-2}$. For $n > 2$ only $(\mathbf{1}, \mathbf{1})_2$ survives, while for $n = 2$ $(\mathbf{1}, \mathbf{1})_{-2}$ survives as well. These states couple to operators of dimension 3 on the boundary which transform as $\mathbf{1}_{\mp 2}$. Recall that the graviton multiplet of $SU(2, 2|2)$ contains an antiself-dual tensor which lives in $\mathbf{1}_2$ of $SU(2)_R \times U(1)_R$. Since the graviton is present for all n , we must identify $\mathbf{1}_2$ 2-form we found above as the superpartner of the graviton. The other 2-form $(\mathbf{1}_{-2})$ which is present only for $n = 2$ then must be a member of some other $SU(2, 2|2)$ multiplet. Such a multiplet is known as a tensor multiplet [27]. Precisely for $n = 2$ the SCFT on the boundary admits a Lagrangian description, so we can identify the operator $\mathbf{1}_{-2}$ couples to as $\sum_{\alpha} \text{Tr}(F_{\alpha} + i\tilde{F}_{\alpha})\Phi_{\alpha}$. The notation here is as follows: F_{α} is the field strength of the α^{th} factor in the gauge group, \tilde{F}_{α} is its dual, Φ_{α} is the complex scalar in the α^{th} vector multiplet. The sum over α arises because we are discussing the bulk sector.

The remaining 2-form $(\mathbf{20}')$ is projected out completely for $n = 2, 4, 6$, while for $n = 3$ a state which transforms as $\mathbf{1}_{-4}$ remains. This 2-form couples to an SCFT operator of dimension 4 which transforms as $\mathbf{1}_4$ with respect to $SU(2)_R \times U(1)_R$. Since we do not have a Lagrangian description of the SCFT for $n = 3$, we cannot write down an explicit form for this operator. However, we will see later that this operator is a superpartner of a Coulomb branch modulus appearing in the Seiberg-Witten curve. Thus its existence follows from the results of section 2 combined with $\mathcal{N} = 2$ supersymmetry.

It remains to analyze the scalar modes. The scalars which couple to marginal or relevant operators have $m^2 \leq 0$ and come in representations $\mathbf{1}_C$, $\mathbf{20}'$, $\mathbf{10}_C$, $\mathbf{50}$, $\mathbf{45}_C$, and $\mathbf{105}$ of $SU(4)$. The easiest to deal with is the massless complex scalar $\mathbf{1}_C$ coming from the reduction of the IIB dilaton. It is a singlet of $SU(4)$, so one could make a hasty conclusion that it remains in the orbifolded spectrum. However, one should not forget that \mathcal{P}_n includes a fractional-linear transformation of the IIB dilaton. For $n = 2$ this is an identity transformation, so $\mathbf{1}_C$ indeed survives. But for $n > 2$ invariance with respect to \mathcal{P}_n requires it to be a root of unity, so $\mathbf{1}_C$ is projected out. This is in perfect agreement with the spectrum of 2-forms found above. Indeed, for $n = 2$ we found an antiself-dual 2-form $\mathbf{1}_{-2}$ which we interpreted as the highest component of the tensor multiplet of $SU(2, 2|2)$ with $m^2 = 1$. The highest component of such a multiplet is a massless complex scalar in $\mathbf{1}_0$. Thus $SU(2, 2|2)$ invariance requires a massless $\mathbf{1}_0$ state. On the other hand for $n > 2$ the 2-form $\mathbf{1}_{-2}$ is projected out, so we do not expect to have any

massless $\mathbf{1}_0$ scalar.

The analysis of the remaining scalars is straightforward. The results are summarized in Tables 1—4. To make tables shorter we only listed the scalars which are primary with respect to $SU(2,2|2)$. In most cases we indicated the type of $SU(2,2|2)$ multiplets they belong to. The graviton multiplet (G) contains a $\mathbf{1}_0$ scalar with $m^2 = -4$, four massless vectors in the adjoint of $SU(2)_R \times U(1)_R$, a $\mathbf{1}_2$ 2-form with $m^2 = 1$ satisfying an equation of the antiself-dual type, and a graviton. (We already saw all the states in this multiplet except the scalar, which arises from $\mathbf{20}'$). The Maxwell multiplet (V) contains a real scalar in $\mathbf{3}_0$ with $m^2 = -4$, a complex scalar in $\mathbf{1}_{-2}$ with $m^2 = -3$, and a massless $\mathbf{1}_0$ vector. There is only one such multiplet in the bulk spectrum: the one containing the $U(1)_L$ gauge boson discussed above. The antiself-dual tensor multiplet T_p contains a complex scalar in $\mathbf{1}_{-2p}$ with $m^2 = p(p-4)$, a real scalar in $\mathbf{3}_{-2p+2}$ with $m^2 = (p+1)(p-3)$, a complex antiself-dual 2-form in $\mathbf{1}_{-2p+2}$ with $m^2 = (p-1)^2$, and a complex scalar in $\mathbf{1}_{-2p+4}$ with $m^2 = p^2 - 4$. The 2-forms we found above fit in T_2 and T_3 , respectively. The tables also contain T_4 ; the corresponding 2-form couples to an irrelevant ($\Delta = 5$) operator and thus did not appear in our analysis.

For $n = 2$ the boundary CFT admits a Lagrangian description in terms of an $Sp(N) \times SU(2N)^{k-1} \times Sp(N)$ gauge theory with two fundamentals for each $Sp(N)$ factor and k bifundamentals. In Table 1 we indicated which gauge theory operators the supergravity states couple to. The notation is as follows: Φ_α is a complex scalar in the $\mathcal{N} = 2$ vector multiplet, $\tilde{Q}_\alpha, Q_\alpha$ are complex scalars in a bifundamental hypermultiplet. The index α runs from 1 to k . Note that we identified the operator coupled to the primary state in the Maxwell multiplet as $\sum_\alpha \tilde{Q}_\alpha Q_\alpha$. This particular linear combination of $\tilde{Q}_\alpha Q_\alpha$ is the only one invariant with respect to the \mathbf{Z}_k subgroup of the orbifold group \mathbf{Z}_{nk} . The supermultiplet containing $\sum_i \tilde{Q}_i Q_i$ also contains $\int d^2\theta \sum_i \tilde{Q}_i Q_i$ which couples to the global mass, and a $U(1)$ current. This $U(1)$ current is the Noether current corresponding to the invariance of the theory with respect to multiplying all Q 's by $e^{-i\phi}$ and all \tilde{Q} 's by $e^{i\phi}$. It follows that the $U(1)_L$ symmetry must be identified with this symmetry of the gauge theory.

In Tables 2—4 we summarize the chiral primaries for \mathbf{Z}_n orbifolds with $n > 2$. The greater is n , the greater is the gap to the next invariant state, therefore the less states we get in the table.

In the preceding analysis we ignored states with nonzero $U(1)_L$ charge. There is a good reason for this. Recall that the \mathbf{Z}_{nk} subgroup of $U(1)_L$ is part

$SU(2, 2 2)$	$SU(2)_R \times U(1)_R$	Δ	SCFT operator
G	$\mathbf{1}_0$	2	
V	$\mathbf{3}_0$	2	$\sum_{\alpha} \text{Tr} \tilde{Q}_{\alpha} Q_{\alpha}$
T_2	$\mathbf{1}_{-4}$	2	$\sum_{\alpha} \text{Tr} \Phi_{\alpha}^2$
T_4	$\mathbf{1}_{-8}$	4	$\sum_{\alpha} \text{Tr} \Phi_{\alpha}^4$
	$\mathbf{3}_{-4}$	4	$\sum_{\alpha} \text{Tr} \tilde{Q}_{\alpha} \Phi_{\alpha}^2 Q_{\alpha}$
	$\mathbf{5}_0$	4	$\sum_{\alpha} \text{Tr} (\tilde{Q}_{\alpha} Q_{\alpha})^2$

Table 1: Bulk spectrum of chiral primaries for $n = 2$. The notation for $SU(2, 2|2)$ multiplets is explained in the text.

$SU(2, 2 2)$	$SU(2)_R \times U(1)_R$	Δ	Comments
G	$\mathbf{1}_0$	2	
V	$\mathbf{3}_0$	2	
T_3	$\mathbf{1}_{-6}$	3	Modulus
	$\mathbf{5}_0$	4	

Table 2: Bulk spectrum of chiral primaries for $n = 3$.

$SU(2, 2 2)$	$SU(2)_R \times U(1)_R$	Δ	Comments
G	$\mathbf{1}_0$	2	
V	$\mathbf{3}_0$	2	
T_4	$\mathbf{1}_{-8}$	4	Modulus
	$\mathbf{5}_0$	4	

Table 3: Bulk spectrum of chiral primaries for $n = 4$.

$SU(2, 2 2)$	$SU(2)_R \times U(1)_R$	Δ
G	$\mathbf{1}_0$	2
V	$\mathbf{3}_0$	2
	$\mathbf{5}_0$	4

Table 4: Bulk spectrum of chiral primaries for $n = 6$.

	$U(1)_L$	$SU(2)_R \times U(1)_R$	Δ	SCFT operator
$n = 2$	2	$\mathbf{3}_0$	2	$\text{Tr}(QJ)^2$
	2	$\mathbf{3}_{-4}$	4	$\text{Tr}(QJ)^2\Phi^2$
	-2	$\mathbf{3}_{-4}$	4	$\text{Tr}\Phi^2(J\tilde{Q})^2$
	2	$\mathbf{5}_0$	4	$\text{Tr}J\tilde{Q}(QJ)^3$
	4	$\mathbf{5}_0$	4	$\text{Tr}(QJ)^4$
$n = 3$	3	$\mathbf{4}_0$	3	
$n = 4$	4	$\mathbf{5}_0$	4	

Table 5: Chiral primaries with nonzero $U(1)_L$ charge for $k = 1$.

of the orbifold group; a \mathbf{Z}_{nk} -invariant state must have $U(1)_L$ charge $L = 0 \bmod nk$. If $L \neq 0$, it must be of order k , therefore for large k all such states come from $SU(4)$ representations with large dimension. Hence for sufficiently large k all states with $L \neq 0$ couple to irrelevant operators. If one is only interested in relevant or marginal operators, one needs to consider states with $L \neq 0$ only for a few low values of k . As an example, we consider the case $k = 1$. The L-charged chiral primaries are shown in Table 5. For $n = 2$ we indicated the operators in the gauge theory the AdS states couple to. As explained above, $U(1)_L$ is a symmetry of the gauge theory with respect to which \tilde{Q} and Q have charges 1 and -1 , while the rest of the fields are neutral. Note also that for $n = 2$, $k = 1$ the SCFT on the boundary has gauge group $Sp(N) \times Sp(N)$, so the bifundamental representation is self-conjugate. The invariant antisymmetric tensors of $Sp(N) \times Sp(N)$ which we denote by J are used to raise and lower gauge indices.

4.2 \mathbf{Z}_{nk} twisted states

The \mathbf{Z}_{nk} orbifold has $nk - 1$ twisted sectors labeled by $j = 1, \dots, nk - 1$. Each twisted sector contributes a $(2, 0)$ tensor multiplet in $d = 6$ [6]. It contains a 2-form with antiself-dual field strength, a complex scalar which is a singlet of $SU(2)_R$, and three real scalars which transform as $\mathbf{3}$ of $SU(2)_R$. Kaluza-Klein (KK) reduction of this tensor multiplet on a circle yields a tower of $SU(2, 2|2)$ multiplets [28]. Let us denote the KK momentum along the circle ℓ . The R-charge of the state is $R = 2\ell$. Recalling the definition of $\mathcal{P}_n J_n$, we infer that the $\mathcal{P}_n J_n$ projection requires the states from the s^{th} sector to have R-charge $-2s \bmod 2n$. The states originating from the complex scalars in $d = 6$ are special because of the additional monodromy: their R-charge is $-2s \pm 2 \bmod 2n$. In either case, since s runs from 1 to $nk - 1$, it is convenient to write it as $s = j + np$, where $j = 0, \dots, n - 1$, while p runs from 1 to $k - 1$ for $j = 0$ and from 0 to $k - 1$ for all other j . (We exclude $j = p = 0$ because it is the bulk sector which we have already analyzed). This parametrization is convenient because it makes obvious that the R-charge mod $2n$ depends only on j , and not on p . Namely, states which do not come from complex scalars have $R = -2j \bmod 2n$. It follows that the multiplicity of such a state is k for $j \neq 0$ and $k - 1$ for $j = 0$. Similarly, the R-charge of the complex scalar states is $\pm 2 - 2j \bmod 2n$.

Let us start with the KK reduction of the 2-forms. In general one expects that a KK reduction of a 2-form in $d = 6$ yields a 2-form on AdS^5 . Such states are always massive [27]. The mass of a 2-form is related to the dimension of an operator on the boundary via $m^2 = (\Delta - 2)^2$. It is easy to see that $m^2 = \ell^2$. The case of zero KK momentum is special: instead of a 2-form KK reduction yields a real massless vector on AdS^5 (see Appendix). As explained above, only states from the $j = 0$ sector are allowed to have zero KK momentum. Their multiplicity is $k - 1$, for any n . Massless vectors couple to conserved currents of dimension $\Delta = 3$. Recalling that we also obtained a $U(1)_L$ gauge boson from the bulk sector, we conclude that the SCFT on the boundary has a total of k $U(1)$ currents. This should be true independent of n .

Next consider the reduction of scalars. According to [28], real $SU(2)$ triplets yield states on AdS^5 with mass $m^2 = \ell^2 - 4$, while complex singlets yield states with $m^2 = \ell^2 \pm 4\ell$. With these formulas at hand we can now find all twisted supergravity states which couple to relevant or marginal operators on the boundary. The results are summarized in Tables 6–8. Again we listed only chiral primaries. Besides the ultra-short vector multiplets which couple

$SU(2, 2 2)$	Multiplicity	$SU(2)_R \times U(1)_R$	Δ	SCFT operator
V	k-1	$\mathbf{3}_0$	2	$\tilde{Q}_\alpha Q_\alpha$
T_2	k	$\mathbf{1}_{-4}$	2	$\text{Tr}\Phi_\alpha^2$
T_3	k-1	$\mathbf{1}_{-6}$	3	$\text{Tr}\Phi_\alpha^3$
T_4	k	$\mathbf{1}_{-8}$	4	$\text{Tr}\Phi_\alpha^4$

Table 6: Chiral primaries from \mathbf{Z}_{nk} -twisted sectors for $n = 2$.

$SU(2, 2 2)$	Multiplicity	$SU(2)_R \times U(1)_R$	Δ	Comments
V	k-1	$\mathbf{3}_0$	2	
T_p	k	$\mathbf{1}_{-2p}, p = 2, 3$	p	Moduli
T_4	k-1	$\mathbf{1}_{-8}$	4	Moduli

Table 7: Chiral primaries from \mathbf{Z}_{nk} -twisted sectors for $n = 3$.

to $U(1)$ currents we also have tensor multiplets $T_{-\ell}$, $\ell < 0$. The lowest component of $T_{-\ell}$ is a scalar with KK momentum ℓ .

4.3 Comparison with Seiberg-Witten curves

In this subsection we compare the supergravity states we found with the results of section 2. First, let us compare the spectrum of $SU(2)_R$ singlet primaries with the Coulomb branch moduli. It was shown in section 2 that the Coulomb branch moduli have R-charge $4, 6, 8, \dots, 2Nn$. The number of

$SU(2, 2 2)$	Multiplicity	$SU(2)_R \times U(1)_R$	Δ	Comments
V	k-1	$\mathbf{3}_0$	2	
T_p	k	$\mathbf{1}_{-2p}, p = 2, 3, 4$	p	Moduli

Table 8: Chiral primaries from \mathbf{Z}_{nk} -twisted sectors for $n = 4$ and $n = 6$.

moduli is determined by their R-charge:

$$\# \text{ of moduli} = \begin{cases} k + 1, & R = 0 \bmod 2n \\ k - 1, & R = 2 \bmod 2n \\ k & \text{otherwise} \end{cases} \quad (21)$$

The dimensions of the moduli are determined by their R-charges as well, $\Delta = R/2$. Since we limited the supegravity analysis to operators with $\Delta \leq 4$, we can only compare the moduli with $R \leq 8$. A quick look at Tables 1—8 shows that indeed all R-charges and multiplicities are in agreement with Eq. (21).

Second, according to section 2, the theories in questions have exactly marginal deformations with zero R-charge corresponding to the locations of the M5' branes and the complex structure of \mathbf{T}^2 . Their number is $k + 1$ for $n = 2$ and k for $n > 2$. We suggest the following identification of these deformations on the supergravity side. According to our tables, for $n = 2$ we have $k + 1$ $SU(2, 2|2)$ multiplets T_2 , while for $n > 2$ we have only k such multiplets. The lowest components of these multiplets couple to operators of dimension 2 and have been identified above as the Coulomb branch moduli. Their highest components are complex scalars which transform as $\mathbf{1}_0$ with respect to $SU(2)_R \times U(1)_R$ and couple to operators of dimension 4. It is natural to interpret these complex scalars as the exactly marginal deformations required by the Seiberg-Witten curve. For $n = 2$ we can check this identification by a gauge theory calculation. In this case the lowest components are simply $\text{Tr} \Phi_\alpha^2$, and the corresponding highest components are $\text{Tr}(F_\alpha^2 + i F_\alpha \tilde{F}_\alpha)$. Deformations by $\text{Tr}(F_\alpha^2 + i F_\alpha \tilde{F}_\alpha)$ are equivalent to changing the gauge couplings and theta-angles. Such deformations are indeed exactly marginal. Furthermore, in the brane construction of section 2 the complexified gauge couplings were encoded in the positions of the M5' branes on \mathbf{T}^2 and the complex structure of \mathbf{T}^2 .

Supergravity also predicts that the SCFT on the boundary has k conserved $U(1)$ currents. One of them couples to a $U(1)_L$ gauge boson from the bulk sector. The other $k - 1$ gauge bosons arise from the twisted sector 2-forms with zero KK momentum (see Appendix). The presence of k $U(1)$ currents is in agreement with the analysis of the Seiberg-Witten curves in section 2. There we identified the mass deformations as the lowest components of the background vector multiplets which couple to conserved currents. In particular, it is natural to identify the global mass as the superpartner of the

$U(1)_L$ current.

The conclusion is that the supergravity spectrum of $SU(2)_R$ singlet scalars matches precisely with the parameters of the Seiberg-Witten curve. The agreement depends crucially on the fact that the F-theory orbifold action contains a nongeometric part J_n .

One interesting feature to note is how the decoupling of $U(1)$'s in the gauge theory is manifested in the supergravity. A signature of decoupling is that the 2-form operators $\text{Tr} F_\alpha$ are absent and one gets global $U(1)$ currents instead. In the supergravity approach a trade-off between a 2-form and a gauge boson occurs because of a subtlety in the KK reduction of the 2-form on $AdS^5 \times S^1$: for zero KK momentum the reduction yields a gauge boson on AdS^5 , while for nonzero KK momentum it yields a massive 2-form. Note also that all masses for the bifundamentals come from the fluxes of the B-fields through the shrunk 2-cycles of $\mathbf{C}^2/\mathbf{Z}_{nk}$. In the case of the global mass parameter it is better to think about an ALF space with a \mathbf{Z}_{nk} singularity rather than about $\mathbf{C}^2/\mathbf{Z}_{nk}$. The global mass arises from the B-field flux through the noncompact 2-cycle of the ALF.

4.4 7-brane excitations

So far we considered states neutral with respect to the 7-brane gauge group H . Let us now discuss the H -charged states. They appear from sectors twisted with respect to $\mathcal{P}_n J_n$ and are localized on 7-branes. For $n = 2$ they are simply open string with both ends of the D7 branes. For $n > 2$ the twisted sector also includes multi-prong strings with ends on mutually nonlocal 7-branes.

To compute the spectrum of 7-brane excitations it is convenient to think of the orbifolding procedure in the following way. One starts with 7-brane states in $d = 8$ with gauge group G . For $n = 2, 3, 4, 6$ $G = Spin(8)/\mathbf{Z}_2, E_6, E_7, E_8$, respectively. All massless states in $d = 8$ live in a vector multiplet in the adjoint of G which includes a vector and a complex scalar. The complex scalar is in the $(\mathbf{1}, \mathbf{1})_2$ of $SU(2)_L \times SU(2)_R \times U(1)_R$. Next one wraps the 7-branes on an orbifold $\mathbf{C}^2/\mathbf{Z}_{nk}$. As explained in section 3, the orbifold group acts nontrivially on the 7-brane gauge bundle and breaks G to a subgroup H . The supergravity spectrum is obtained by reducing the $d = 8$ spectrum to $d = 5$, decomposing the adjoint of G into representations of H , and then projecting on the states invariant with respect to the \mathbf{Z}_{nk} action. The resulting $d = 5$ spectrum depends on the manner in which G is broken down

to H . Since we know H only for $n = 2$, we first investigate this case.

For $n = 2$ the orbifold action on G is given by Eq. (18). The unbroken gauge group is $H = (Spin(4) \times Spin(4))/\mathbf{Z}_2$. The adjoint of $G = Spin(8)/\mathbf{Z}_2$ decomposes into an adjoint and a pair of $(\mathbf{4}, \mathbf{4})$ representations of H . The adjoint of H commutes with \mathcal{M} , while the $(\mathbf{4}, \mathbf{4})$ representations anticommute with it. As explained in [25], the reduction of the $d = 8$ vector multiplet to $d = 5$ produces a tower of vector multiplets whose lowest components are real scalars in $(\mathbf{p}, \mathbf{p} + \mathbf{2})_0$ of $SU(2)_L \times SU(2)_R \times U(1)_R$, and $p = 1, 2, 3, \dots$. The scalars are chiral primary states, so their masses are given by $m^2 = (p + 1)(p - 3)$. For $p = 1$ the vector multiplet contains a massless vector, while for $p > 1$ it contains a massive vector. The requirement that the scalar couple to a relevant or marginal operator on the boundary restricts p to be 1, 2, or 3.

To perform the \mathbf{Z}_{2k} projection we recall that the geometric part of \mathbf{Z}_{2k} is embedded into $U(1)_L \subset SU(2)_L$. First let us look at states with zero $U(1)_L$ charge L . Such states are present only for odd p . For these states only the nongeometric part of \mathbf{Z}_{2k} is nontrivial. This means that the $(\mathbf{4}, \mathbf{4})$ representation of H is projected out and only the adjoint of H remains.

Turning now to states with $L \neq 0$, it is easy to see that unless $k = 1$ or 2 they only couple to irrelevant operators. Indeed, invariance with respect to the \mathbf{Z}_k subgroup of \mathbf{Z}_{2k} requires that L be a multiple of k . But the states which couple to relevant or marginal operators (i.e. states with $p = 1, 2, 3$) have $|L| \leq 2$. Thus we only need to consider $k = 1$ and $k = 2$.

For $k = 1$ we need to perform a \mathbf{Z}_2 projection. The generator of \mathbf{Z}_2 is a product of a rotation by π in $U(1)_L$ and a conjugation by \mathcal{M} . The $p = 2$ scalar transforms as $(\mathbf{2}, \mathbf{4})_0$ of $SU(2)_L \times SU(2)_R \times U(1)_R$, so it is odd with respect to the π rotation. Hence its \mathbf{Z}_2 -invariant part is a complex scalar in the $(\mathbf{4}, \mathbf{4})$ of H which transforms as $\mathbf{4}_{0,+2}$ of $SU(2)_R \times U(1)_R \times U(1)_L$. The $p = 3$ scalar transforms as $(\mathbf{3}, \mathbf{5})_0$ of $SU(2)_L \times SU(2)_R \times U(1)_R$, and its \mathbf{Z}_2 -invariant part includes a complex scalar in the adjoint of H and in $\mathbf{5}_{0,+2}$ of $SU(2)_R \times U(1)_R \times U(1)_L$. (It also includes a $\mathbf{5}_0$ scalar with $L = 0$ which we have already discussed.)

For $k = 2$ we need to perform a \mathbf{Z}_4 projection. The generator of \mathbf{Z}_4 is a product of a rotation by $\pi/2$ in $U(1)_L$ and a conjugation by \mathcal{M} . The $p = 2$ scalar is projected out completely, while the $p = 3$ scalar yields a complex scalar in the $(\mathbf{4}, \mathbf{4})$ of H and in $\mathbf{5}_{0,+2}$ of $SU(2)_R \times U(1)_R \times U(1)_L$.

The results of this analysis are summarized in Table 9. We matched all the states we found with gauge theory operators. The gauge group for

H	$U(1)_L$	$SU(2)_R \times U(1)_R$	Δ	SCFT operator	Comments
Adjoint	0	$\mathbf{3}_0$	2	$\tilde{q}_\sigma q_\sigma$	
Adjoint	0	$\mathbf{5}_0$	4	$\tilde{q}_\sigma Q \tilde{Q} q_\sigma$	
(4,4)	1	$\mathbf{4}_0$	3	$\tilde{q}_2 J Q J q_1$	$k = 1$
Adjoint	2	$\mathbf{5}_0$	4	$\tilde{q}_\sigma (Q J)^2 q_\sigma$	$k = 1$
(4,4)	2	$\mathbf{5}_0$	4	$\tilde{q}_2 Q_2 Q_1 J q_1$	$k = 2$

Table 9: States charged with respect to the 7-brane gauge group H for $n = 2$. Some of the states are present only for special values of k ; this is indicated in the last column.

$n = 2$ contains two symplectic factors $Sp(N)_1$ and $Sp(N)_2$ with two fundamentals for each factor. In Table 9 $q_\sigma, \tilde{q}_\sigma, \sigma = 1, 2$, denote the scalars in the fundamental hypermultiplet of $Sp(N)_\sigma$.

Finally let us discuss the 7-brane states for $n = 3, 4, 6$. The analysis of states with zero $U(1)_L$ charge proceeds in exactly the same way as for $n = 2$ and produces the same spectrum of states. All these states live in the adjoint of H . The only difference from the case $n = 2$ is that now we do not know the monodromy \mathcal{M} , and so do not know H . Again it is easy to see that for $k > 2$ the states with $L \neq 0$ couple only to irrelevant operators. To analyze the spectrum of states with $L \neq 0$ for $k = 1, 2$ one needs to know \mathcal{M} .

5 Conclusions and outlook

In this paper we constructed and studied a class of $\mathcal{N} = 2$ superconformal field theories in four dimensions. These theories are labeled by two integers n and k . n takes values 2, 3, 4, 6, and k is an arbitrary positive integer. For $n = 2$ the model in question is a finite $\mathcal{N} = 2$ gauge theory with gauge group $Sp(N) \times SU(2N)^{k-1} \times Sp(N)$. For $n > 2$ the models are new $\mathcal{N} = 2$ field theories which do not admit a Lagrangian description. Using M and F-theory methods we learned quite a lot about these theories: we determined the Seiberg-Witten curve and found the spectrum of operators in short representations of the superconformal group. The M-theory approach allows to study the theories for finite N but is limited to superconformal families containing the Coulomb branch moduli. To learn about other families we

used F-theory and the AdS/CFT correspondence. This approach is effective for large N . The F-theory construction turns out to be quite intricate and involves frozen $\mathbf{C}^2/\mathbf{Z}_n$ singularities. It is satisfying to see that whenever both methods apply they give identical results. In particular, we showed that the decoupling of $U(1)$'s in the gauge theory on the boundary of AdS^5 is due to a subtlety in the KK reduction of a 2-form on $AdS^5 \times S^1$.

An interesting direction to pursue is to study in more detail F-theory backgrounds with frozen orbifold singularities. Only in the case $n = 2$, where the background can be understood as an orientifold background in IIB, do we have a complete control over the twisted sectors of the orbifold. For $n > 2$ the twisted sectors must include multi-prong strings. It would be quite interesting to study the structure of the twisted sectors in detail and in particular determine the gauge group living at the singularity. It is likely that this can be done along the lines of [29].

One could extend our results by considering 7-branes wrapping orbifold singularities other than A_n . For example, one could consider D7-branes wrapping a D_n -type singularity. In M-theory this corresponds to adding “NS orientifolds” [30] parallel to NS5 branes. The Seiberg-Witten solution for the probe theory will again be encoded in a Hitchin system on an orbifolded torus, the main difference being that the gauge group of Hitchin equations will be $SO(2N)$ rather than $U(2N)$.

Appendix: Kaluza-Klein reduction of 2-form on $AdS^5 \times S^1$

Consider a 2-form B on $AdS^5 \times S^1$ whose field strength $G = dB$ satisfies $G = - * G$. We are going to show that the Kaluza-Klein reduction of B produces either a massless vector on AdS^5 satisfying Maxwell equations, or a massive 2-form satisfying equations of the anti-selfdual type, depending on whether the momentum along S^1 is zero or not.

The Kaluza-Klein ansatz is

$$B = (a \wedge dt + b)e^{it\ell}, \quad (22)$$

where a is a 1-form on AdS^5 , b is a 2-form on AdS^5 , and ℓ is the integer-valued KK momentum. Then G and $*G$ are given by

$$G = (da \wedge dt + db + i\ell b \wedge dt)e^{it\ell}, \quad *G = (*da + *db \wedge dt + i\ell *b)e^{it\ell}. \quad (23)$$

The equation $G = - * G$ implies an equation of motion for a and b :

$$da + i\ell b = - * db. \quad (24)$$

The field B in $d = 6$ has a gauge invariance $B \rightarrow B + d\Lambda$, where Λ is a 1-form. Kaluza-Klein ansatz is invariant with respect to a subset of these gauge transformations, namely those with Λ of the form

$$\Lambda = (\sigma dt + \lambda) e^{it\ell}. \quad (25)$$

Here σ and λ are 0- and 1-form on AdS^5 , respectively. The induced transformations on a and b are

$$a \rightarrow a + d\sigma + i\ell\lambda, \quad b \rightarrow b + d\lambda. \quad (26)$$

When $\ell = 0$ the gauge invariance for a is the standard gauge invariance for the massless vector field, $a \rightarrow a + d\sigma$. The equation of motion takes the form $da = - * db$. This is equivalent to Maxwell equations $d * da = 0$. Then b is not an independent field: it is determined by a up to a gauge transformation.

When $\ell \neq 0$ gauge freedom can be used to set $a = 0$. Then b does not have any residual gauge invariance. Its equation of motion becomes $i\ell b = - * db$. This is an equation of the anti-selfdual type describing a massive 2-form on AdS^5 [26, 27].

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